JHEP HYPER VERSION

Received: *May 13, 2001*, Accepted: *June 27, 2001*

Graded lagrangians, exotic topological D-branes and enhanced triangulated categories

Calin Iuliu Lazaroiu

C.N. Yang Institute for Theoretical Physics
SUNY at Stony BrookNY11794-3840, USA
E-mail: `calin@insti.physics.sunysb.edu`

Abstract: I point out that (BPS saturated) A-type D-branes in superstring compactification on Calabi-Yau threefolds correspond to *graded* special lagrangian submanifolds, a particular case of the graded lagrangian submanifolds considered by M. Kontsevich and P. Seidel. Combining this with the categorical formulation of cubic string field theory in the presence of D-branes, I consider a collection of *topological* D-branes wrapped over the same lagrangian cycle and *derive* its string field action from first string-theoretic principles. The result is a **Z**-*graded* version of super-Chern-Simons field theory living on the lagrangian cycle, whose relevant string field is a degree one superconnection in a **Z**-graded superbundle, in the sense previously considered in mathematical work of J.M. Bismutt and J. Lott. This gives a refined (and modified) version of a proposal previously made by C. Vafa. I analyze the vacuum deformations of this theory and relate them to topological D-brane composite formation, upon using the general formalism developed in a previous paper. This allows me to identify a large class of topological D-brane composites (generalized, or "exotic" topological D-branes) which do not admit a traditional description. Among these are objects which correspond to the "covariantly constant sequences of flat bundles" considered by Bismut and Lott, as well as more general structures, which are related to the enhanced triangulated categories of Bondal and Kapranov. I also give a rough sketch of the relation between this construction and the large radius limit of a certain version of the "derived category of Fukaya's category". This paper forms part of a joint project with Prof. S. Popescu, a brief announcement of which can be found in the second part of the note `hep-th/0102183`. The paralel B-model realization, as well as the relation with the enhanced triangulated categories of Bondal and Kapranov, was recently discussed by D.E. Diaconescu in the paper `hep-th/0104200`, upon using the observations contained in that announcement.

Keywords: Chern-Simons Theories, D-branes, Superstring Vacua.

Contents

1. Introduction

Recent progress in our understanding of Calabi-Yau D-brane physics has lead to the realization that the D-branes of an $N = 2$ superstring compactification are intrinsically graded objects. This observation, which in the physics context is due to M. Douglas [3, 4], and in the homological mirror symmetry literature can be traced back to [1] (see also [2] and [6]), has far reaching implications for the physical description of open superstring dynamics on Calabi-Yau backgrounds. Most work toward extracting the physical consequences of this fact [3, 5, 7, 16], has focused on the analysis of so-called B-type branes [9], i.e. those D-branes which in the large radius limit are described by holomorphic sheaves. The other class of D-branes which can be introduced in such compactifications (the so-called A-branes, which correspond to (special) lagrangian cycles) has been comparatively less well studied. This asks for clarification, especially since applications to mirror symmetry (which interchanges the two types of branes) require that we understand both sides of the mirror duality.

Since the chiral/antichiral primary sectors of the compactified superstring are faithfully described by the associated topological models [26, 27, 28], it is natural to approach this problem from the point of view of topological string theory. The purpose of the present paper is to analyze some basic dynamical implications of the existence of D-brane grading for the topological branes of the A model.

Most recent work on categories of D-branes on Calabi-Yau manifolds follows the largely *on-shell* (or rather, partially off-shell) approach originally proposed in [3]. In [13], I proposed an alternative, *consistently off-shell* approach, which makes direct use of a formalism (developed in [12]) for associative string field theory in the presence of D-branes and goes well-beyond the boundary state formalism [25, 34] which has traditionally dominated the subject.[1] This method is especially well-suited for analysis of D-brane composite formation, a phenomenon which forms the crucial ingredient of any realistic approach to Calabi-Yau D-brane physics. Since condensation of boundary and boundary condition changing operators is intrinsically an *off-shell* process, it can be expected that a complete understanding of such phenomena can only be obtained by systematic use of off-shell techniques, which explains the relevance of string field theory methods. This point of view also allows for better contact with the homological mirror symmetry conjecture of [1], which is based on an effort to work at the (co)chain, rather than (co)homology level.

In this paper I shall follow the approach outlined in [13] by giving its concrete realization for a class of topological D-branes of the A model. A thorough understanding of A-model open string dynamics requires consideration of intersecting D-branes and a detailed analysis of disk instanton effects, which ultimately should be carried out through mirror symmetry. Since a complete discussion of these issues (which bear an intimate relation with the mathematical work of K. Fukaya [35, 36]) is rather involved, the present paper restricts to a small sector of the relevant structure, by considering a system of D-branes which wrap a given (special) lagrangian cycle. We shall moreover take the large radius limit, thereby neglecting disk instan-

[1]Some limitations of the (topological) boundary state formalism were discussed in [10].

ton corrections [11] to the classical string field action. As we shall see below, even the analysis of this sector is considerably more subtle than has previously been thought. In fact, insistence on a consistent off-shell description leads to novel results, which would be difficult to extract through the "mixed" methods of [3, 5]. Among these is is the fact that the string field associated with such boundary sectors is a superconnection living in a $\mathbf{Z}$-*graded* superbundle, rather that the standard $\mathbf{Z}_2$-graded variant of [18] which was used in [8, 20]. More importantly, we shall *show the existence of many more classes of* topological *A-type branes than has previously been suspected*. Such exotic topological branes correspond to backgrounds in the *extended* moduli space of the open topological A-string. Whether they also play a role in the physical, untwisted theory is a question which we do not attempt to settle in this paper.

The paper is organized as follows. In section 2, we recall the basics of the general formalism of associative open string field theory with D-branes which was developed in [12, 13]. Since we shall later deal with the A-model, which for certain D-brane configurations admits a "complex conjugation" symmetry, we also give a brief discussion of the supplementary structure describing open string field theories endowed with such operations. This is a straightforward extension of the analysis of [12], which bears on basic structural issues such as consistent general constructions of antibranes. It can be viewed as a D-brane extension of the analysis already given in [29, 30, 33], though our discussion is carried in different conventions. In section 3 I reconsider the problem of grading of A-type topological D-branes. While this issue was touched upon in [3] (see also [5]), a careful geometric analysis does not seem to have been given before. I give a precise description of this grading and discuss some of the underlying issues of orientation, which turn out to be of crucial importance for the string field theory discussed in later sections. Our approach makes us of the so-called *graded lagrangian submanifolds* of [2], which can be shown to give the general description of (not necessarily BPS saturated) graded topological A-branes. Since the present paper restricts to the special lagrangian case, I only discuss this theory in its very simplified form which applies to such situations. This gives a geometrically self-contained description of A-type branes as graded objects, thereby improving on the discussion of [3, 5]. The general theory, as it applies to non-BPS A-type branes, will be discussed somewhere else [15]. Armed with a consistent off-shell framework and a precise understanding of the geometric description of our objects, we proceed in section 4 to construct the string field theory of a system of *distinct* D-branes wrapping the same special lagrangian cycle. More precisely, we consider a set of branes having different gradings and whose underlying geometric supports coincide. The fact that one must consider such systems is a direct consequence of the idea of D-branes as graded objects, and it should serve as a test for its implications. The salient point of our construction is that the presence of distinct gradings leads to a shift of the worldsheet U(1) charge of the various boundary condition changing observables, which implies that the resulting string field theory is a **Z**-*graded* version of super-Chern-Simons theory.[2] This suggests that the currently prevalent approach (which is largely based on borrowing the results of [26]) must be reconsidered. We build the string field theory by identifying each piece of the axiomatic data discussed in [12, 13], and check the relevant consistency constraints. This gives an explicit realization of the general framework developed in those papers. After identifying the underlying structure, we show that the resulting string field action admits a description in the language of **Z**-*graded superconnections*, which were previously considered in the mathematical work of [19]. We proceed in section 5 with a discussion of the conditions under which our string field theory is invariant with respect to complex conjugation, and give a precise description of the conjugation operators. Section 6 formulates an extended string field action, whose detailed analysis is left for future work. Section 7 gives a preliminary analysis of the moduli space of vacua of our string field theory. Upon applying the general framework of [12, 13], we discuss the relevant deformation problem and sketch its relation with the modern mathematical theory of *extended* deformations [21, 22]. This provides a concrete realization of some general observations made in [13]. In section 8 we analyze various types of deformations, which allows us to identify large classes of topological A-type branes (wrapping the cycle L) which *do not admit a traditional description.* These "exotic A-type branes" can be viewed as topological D-brane composites resulting from condensation of boundary and boundary condition changing operators, and they must be included in a physically complete analysis of A-type open string dynamics. As a particular case, we recover the standard deformations of traditional A-type branes, and a class of generalized D-branes which correspond to the flat complexes of vector bundles studied in [19]. We also discuss more general solutions, which correspond to the pseudocomplexes and generalized complexes of [12], and relate the former to the enhanced triangulated categories of Bondal and Kapranov, upon following the general observations already made in [13]. This provides a vast enlargement of the theory of [26], which can be analyzed through the categorical methods developed in [12]. Section 9 makes some brief remarks on how the theory considered in this paper relates to Fukaya's category [35, 36]. Finally, section 10 presents our conclusions.

The content of this paper was originally conceived as introductory material for the more detailed work [14], a brief announcement of which can be found in the last section of the note [13]. Meanwhile, a paper appeared [16], which succeeded to recover some of the B-model details which were left out in [13], as well as the specifics of the relation with the work of Bondal and Kapranov [17], a relation which was mentioned in [13]. This prompted me to write the present note, which contains a self-contained description of (part) of the A-model realization of the relevant string

[2]This should be compared with the $\mathbf{Z}_2$-graded proposal of [20].

JHEP06(2001)064

field theory, and a brief sketch of the ensuing mathematical analysis. Since the basic B-model realization is now available in [16], I refer the reader to this reference for the parallel holomorphic discussion.[3]

2. Review of axioms and complex conjugation

We start with a short review of the necessary framework, followed by a brief discussion of complex conjugations. We are interested in a general description of the structure of cubic (or associative, as opposed to homotopy associative) open string field theory in the presence of D-branes. This subject was discussed systematically in the paper [12], to which we refer the reader for details. The basic idea is to formulate the theory in terms of a so-called "differential graded (dG) category", which encodes the spaces of off-shell states of strings stretching between a collection of D-branes. Since we are interested in oriented strings, such states can be viewed as morphisms between the various D-branes, which allows us to build a category with objects given by the D-branes themselves. The associative morphism compositions are given by the basic string products, which result from the triple correlators on the disk. From this perspective, states of strings whose endpoints lie on the same D-brane a define "diagonal" boundary sectors $\mathrm{Hom}(a,a) = \mathrm{End}(a)$, while states of strings stretching between two different D-branes a and b (namely from a to b) define boundary condition changing sectors $\mathrm{Hom}(a,b)$. This terminology is inspired by the formalism of open-closed conformal field theory on surfaces with boundary [25]; indeed, states in the boundary and boundary condition changing sectors are related to boundary/boundary condition changing conformal field theory operators via the state-operator correspondence. The reader is referred to [13] for a non-technical introduction to this approach and to [12] for a more detailed discussion. Some background on dG categories can be found in the appendix of [12], while a systematic discussion of the relevant on-shell approach can be found in [10].

2.1 The basic data and the string field action

Recall from [12] that a (tree level) associative open string field theory in the presence of D-branes is specified by:

(I) A differential graded $\mathbf{C}$-linear category $\mathcal{A}$

(II) For each pair of objects a, b of $\mathcal{A}$, an invariant non-degenerate bilinear and graded-symmetric form ${}_{ab}(.,.)_{ba} : \mathrm{Hom}(a,b) \times \mathrm{Hom}(b,a) \to \mathbf{C}$ of degree 3.

[3] I should perhaps point out that the idea of using the work of Bondal and Kapranov [17] goes back to the original paper of M. Kontsevich on the homological mirror symmetry conjecture. This idea was re-iterated in [13] in a string field theory context, where its application to the B model was mentioned without giving all of the relevant details (which form part of the more ambitious project [14]). The paper [16] follows this idea by studying a $\mathbf{Z}$-graded holomorphic theory, which was implicit (but not explicitly written down) in the announcement [13]. That theory, as well as the theory of this paper, can be obtained through the very general procedure of *shift-completion*.

JHEP06(2001)064

A $\mathbf{C}$-*linear category* is a category whose morphism spaces are complex vector spaces and whose morphism compositions are bilinear maps. A *graded linear category* is a linear category whose morphism spaces are $\mathbf{Z}$-graded, i.e. $\mathrm{Hom}(a,b) = \oplus_{k\in\mathbf{Z}}\mathrm{Hom}^k(a,b)$, and such that morphism compositions are homogeneous of degree zero, i.e.

$$|uv| = |u| + |v|\,, \qquad \text{for all homogeneous } u \in \mathrm{Hom}(b,c)\,, \qquad v \in \mathrm{Hom}(a,b)\,, \tag{2.1}$$

where $|.|$ denotes the degree of a homogeneous element. In a *differential graded linear category* (dG category), the morphism spaces $\mathrm{Hom}(a,b)$ are endowed with nilpotent operators Q_{ab} of degree $+1$ which act as derivations of morphism compositions:

$$Q_{ac}(uv) = Q_{bc}(u)v + (-1)^{|u|}uQ_{ab}v\,, \qquad \text{for homogeneous } u \in \mathrm{Hom}(b,c)\,, \quad v \in \mathrm{Hom}(a,b)\,. \tag{2.2}$$

Graded symmetry of the bilinear forms means:

$${}_{ab}(u,v)_{ba} = (-1)^{|v||u|}\,{}_{ba}(v,u)_{ab} \tag{2.3}$$

for homogeneous elements $u \in \mathrm{Hom}(a,b)$ and $v \in \mathrm{Hom}(b,a)$. The degree 3 constraint is:

$${}_{ab}(u,v)_{ba} = 0\,, \qquad \text{unless } |u| + |v| = 3\,. \tag{2.4}$$

The bilinear forms are required to be invariant with respect to the action of Q_{ab} and morphism compositions:

$${}_{ab}(Q_{ab}(u),v)_{ba} + (-1)^{|u|}(u, Q_{ba}(v)) = 0\,, \qquad \text{for } u \in \mathrm{Hom}(a,b)\,, \quad v \in \mathrm{Hom}(b,a) \tag{2.5}$$

and

$${}_{ca}(u,vw)_{ac} = {}_{ba}(uv,w)_{ab}\,, \qquad \text{for } u \in \mathrm{Hom}(c,a)\,, \quad v \in \mathrm{Hom}(b,c)\,, \quad w \in \mathrm{Hom}(a,b)\,. \tag{2.6}$$

Given such data, one can build the (unextended) string field action:

$$S(\phi) = \frac{1}{2}\sum_{a,b\in S} {}_{ba}\langle\phi_{ba}, Q_{ab}\phi_{ab}\rangle_{ab} + \frac{1}{3}\sum_{a,b,c\in S} {}_{ca}\langle\phi_{ca}, \phi_{bc}\cdot\phi_{ab}\rangle_{ac}\,, \tag{2.7}$$

where S denotes the set of objects of $\mathcal{A}$[4] and $\phi_{ab} \in \mathrm{Hom}^1(a,b)$ are the components of the degree one string field.

[4]We assume for simplicity that $\mathcal{A}$ is a small category, i.e. its objects form a set. In fact, we shall only need the case when S is finite or countable, since this will be relevant for our application. We neglect the conditions under which various sums make mathematical sense (i.e. converge etc). Convergence of these sums can be assured (at least in principle) by imposing supplementary conditions on the allowed string field configurations.

JHEP06(2001)064

Upon defining the total boundary space $\mathcal{H} = \oplus_{a,b\in S}\mathrm{Hom}(a,b)$, the total string product $\cdot : \mathcal{H}\times\mathcal{H}\to\mathcal{H}$ as well as the total bilinear form $\langle .,.\rangle$ and BRST operator $Q:\mathcal{H}\to\mathcal{H}$ in an obvious manner (see [12] for details), one can write this action in the more compact form:

$$S(\phi) = \frac{1}{2}\langle\phi, Q\phi\rangle + \frac{1}{3}\langle\phi, \phi\cdot\phi\rangle\,, \tag{2.8}$$

where $\phi = \oplus_{a,b\in S}\phi_{ab}$ is a degree one element of $\mathcal{H}$.

2.2 Theories with complex conjugation

The axioms of a string field theory can be supplemented by requiring the existence of conjugation operators subject to certain constraints.[5] We say that such a theory is *endowed with conjugations* if one is given the following two supplementary structures:

(IIIa) An involutive map $a \to \overline{a}$ on the set of D-brane labels.

(IIIb) A system of *degree zero* antilinear operators $*_{ab} : \mathrm{Hom}(a,b) \to \mathrm{Hom}(\overline{b},\overline{a})$, with the properties:

(1) $*_{\overline{a}\overline{b}} *_{ba} = id_{\mathrm{Hom}(b,a)}$, for any two objects a and b

(2) ${}_{\overline{b}\overline{a}}\langle *_{ab}u, *_{ba}v\rangle_{\overline{a}\overline{b}} ={}_{ba}\overline{\langle v,u\rangle}_{ab}$ for $u\in\mathrm{Hom}(a,b)$ and $v\in\mathrm{Hom}(b,a)$

(3) $Q_{\overline{b}\overline{a}} *_{ab} u = (-1)^{|u|+1} *_{ab} Q_{ab}u$ for $u \in \mathrm{Hom}(a,b)$

(4) $*_{ac}(u_{bc}v_{ab}) = (*_{ab}v_{ab})(*_{bc}u_{bc})$ for $u_{bc}\in\mathrm{Hom}(b,c)$ and $v_{ab}\in\mathrm{Hom}(a,b)$.

For each D-brane a, its partner $\overline{a}$ will be called its "conjugate brane".

In a theory possessing conjugations, the action (2.8) has the following property:

$$\overline{S(\phi)} = S(*\phi)\,. \tag{2.9}$$

Hence one can assure reality of the string field action by imposing the following condition on the string field:

$$*\phi = \phi \Longleftrightarrow *_{ab}\phi_{ab} = \phi_{\overline{b}\overline{a}}\,. \tag{2.10}$$

The condition that $*$ preserve the degree $|.|$ is crucial for consistency of the reality constraint $|*\phi| = |\phi|$ with the degree one constraint $|\phi| = 1$.

[5] This type of structure was discussed in [33] for the case of a single boundary sector (but allowing for homotopy associative string products).

3. A-type D-branes as graded special lagrangian submanifolds

It is well known [9] that BPS saturated D-branes in Calabi-Yau threefold compactifications are described either by holomorphic cycles (so-called type B-branes) or by special lagrangian cycles (so-called type A D-branes) of the Calabi-Yau target space X. In the case of multiply-wrapped branes one must also include a bundle living on each cycle (which corresponds to a choice of Chan-Paton data) and a choice of connection in this bundle, which should be integrable for B-type branes and flat for type A branes. From a conformal field theory point of view, type A and B D-branes correspond to different boundary conditions [9], and they preserve different $N = 2$ subalgebras of the $(2, 2)$ worldsheet algebra.

The starting point of our analysis is the observation of [3] that this data does not in fact suffice for a complete description of D-brane physics. The essence of the argument of [3] is as follows. The various boundary/boundary condition changing sectors of the theory consist of open string states, which are charged with respect to the worldsheet U(1) current of the $N = 2$ superconformal algebra. It is then shown in [3] (based on bosonization techniques) that a complete specification of the theory requires a consistent choice of charges for the various boundary sectors, and that, in the presence of at least two different D-branes, the relative assignment of such charges has an invariant physical meaning. If one defines the "abstract" degree of a D-brane through the winding number of the bosonized U(1) current, this amounts to the statement that a complete description of the background requires the specification of an integer number for each D-brane present in the compactification.

While this is an extremely general argument (which in particular applies to non-geometric compactifications), the concrete realization for the A model remains somewhat obscure. In this section, I explain the geometric meaning of this "grade" for the case of semiclassical type A branes, i.e. for the large radius limit of a Calabi-Yau compactification which includes such objects.[6] Our proposal is motivated by a combination of mathematical results of [2] and an anomaly analysis which can be carried out in the twisted model (the A-model). Since a complete exposition would take more space than afforded in this paper, I will give a simplified discussion and refer the reader to [15], which will include a more detailed analysis.

We propose that the correct description of a BPS saturated type A brane is given (in the large radius limit) by a triple $(\mathbf{L}, E, A)$, where $\mathbf{L}$ is a so-called *graded special lagrangian submanifold* of X. The mathematical concept of graded lagrangian manifolds (not necessarily special) is originally due to M. Kontsevich [1] and was

[6]If one wishes to go away from large radius, then one must consider disk instanton corrections to the boundary/boundary condition changing sectors, which are induced by corrections to the associated BRST operators. This leads to Floer cohomology and instanton destabilization of some semiclassical D-branes, as well as to supplementary corrections to the string field action. By staying in the large radius limit, we avoid each of these issues.

discussed in more detail in recent work of P. Seidel [2]. Recall that a lagrangian cycle is a three-dimensional submanifold of Xsuch that the Kahler form ω of X has vanishing restriction to L. The cycle is *special* lagrangian, if also also has:

$$\mathrm{Im}(\lambda_L \Omega)|_L = 0 \tag{3.1}$$

for some complex number λ_L of unit modulus. Since this condition is invariant under the change $\lambda_L \to -\lambda_L$, the relevant quantity is λ_L^2, i.e. the special lagrangian condition (3.1) only specifies λ_L *up to sign.* A *grading*[7] of a special lagrangian cycle L is simply the choice of a *real* number ϕ_L such that $e^{-2\pi i \phi_L} = \lambda_L^2$. There is a discrete infinity of such choices (differing by an integer), so there is a countable infinity of gradings of any given special lagrangian cycle. Furthermore, there is always a *canonical* choice $\phi_L^{(0)} \in [0, 1)$ (which we shall call the *fundamental grading*), a fact which distinguishes special lagrangians from more general lagrangian submanifolds. Then any grading is of the form $\phi_L^{(n)} = \phi_L^{(0)} + n$, with n an integer which specifies a shift of the charges of the associated boundary sector. Hence one can identify a graded special lagrangian cycle $\mathbf{L}$ with the pair (L, n). This agrees with the general conformal field theory argument of [3].

We end this section by noting that a choice of grading specifies an orientation of the special lagrangian cycle. Indeed, given a grading $\phi_L^{(n)}$, one has a canonically-defined *real* volume form $\Omega_L(n) := e^{-i\pi\phi_L^{(n)}}\Omega = e^{-i(\phi_L^{(0)}+n)}\Omega$ on L.[8] It is clear that the orientation induced by $\phi_L^{(n)} = \phi_L^{(0)} + n$ in this manner coincides with $(-1)^n$ times the orientation induced by the fundamental grading $\phi_L^{(0)}$.

4. The open string field theory of a collection of graded D-branes wrapping the same special lagrangian cycle

We consider a family of D-branes a_n described by triples $(\mathbf{L}_n, E_n, A_n)$ with $\mathbf{L}_n := (L, n)$ for some collection of integers n. These branes share the same underlying special lagrangian cycle L, but have possibly different Chan-Patton bundles E_n (complex vector bundles defined over the cycle L) and different background flat connections A_n living in these bundles. Note that we assume that each brane has a different grade n.

[7]It is not hard to show [15] that this is a particular case of the general notion of graded lagrangian submanifold introduced in [2].

[8]One can also think of this as a choice of sign, $\lambda_L(n) := e^{-i\pi\phi_L(n)}$, for the quantity appearing in (3.1) (the other choice afforded by the relation $\lambda_L^2 = e^{-2\pi i\phi_L^{(n)}}$ would be $\lambda'_L(n) = -\lambda_L = e^{-i\pi\phi_L^{(n+1)}}$, which corresponds to the opposite orientation). The point is that once one specifies a grading n, we have a choice $\lambda_L(n)$ which is uniquely determined by n, and consequently we have a natural choice of orientation. In the absence of a grading, we would have no natural way to pick one of the two opposite orientations. This may seem subtle but it is in fact a triviality. As we shall see below, this simple fact is ultimately responsible for the appearance of supertraces in our string field action.

JHEP06(2001)064

According to our previous discussion, each D-brane a_n determines an orientation $\mathcal{O}_n$ of the cycle L, and these orientations are related to the "fundamental" orientation $\mathcal{O}_0$ through:

$$\mathcal{O}_n = (-1)^n \mathcal{O}_0 \,. \tag{4.1}$$

When integrating differential forms (see below), we shall write L_n for L endowed with the orientation $\mathcal{O}_n$, and $L := L_0$ for L endowed with the orientation $\mathcal{O}_0$.

We now consider the cubic open string field theory associated with such a system. Since we have more than one D-brane, one must use the categorical framework of [12, 13], which was shortly reviewed in section 2. Instead of giving a lengthy discussion of localization starting from the topological A model, we shall simply list the relevant data and check that the axioms of section 2 are satisfied. The data of interest is as follows:

(1) The spaces $\mathrm{Hom}(a_m, a_n)$ of off-shell states of oriented open strings stretching from a_m to a_n. For our topological field theory, these can be identified through a slight extension of the arguments of [26], which gives:

$$\mathrm{Hom}^k(a_m, a_n) = \Omega^{k+m-n}(L, \mathrm{Hom}(E_m, E_n))\,, \tag{4.2}$$

where $\Omega^*(L, \mathrm{Hom}(E_m, E_n))$ denotes the space of (smooth) differential forms on the cycle L with values in the bundle $\mathrm{Hom}(E_m, E_n)$. In these expressions we let k be any signed integer, so for example $\mathrm{Hom}^k(a_m, a_n)$ can be non vanishing for $k = n - m \cdots n - m + 3$ and vanishes otherwise.

In the string field theory of the A model, the various state spaces must graded by the charge of boundary/ boundary condition changing states with respect to the anomalous U(1) current on the string worldsheet (this follows from the construction of the string field products in the manner of [31, 32]). In our situation, this grading (which is indicated by the superscript k in (4.2)) differs from the grading by form rank, as displayed by the shift through $m - n$ in (4.2). The presence of such a shift follows from arguments similar to those of [3], or by a careful discussion of localization along the lines of [26]. These shifts of the U(1) charge in the boundary condition changing sectors reflect the different gradings of the branes a_n and are required for a consistent description of the theory. They will play a crucial role in the correct identification of the string field theory of our D-brane system. If one denotes the U(1) charge of a state $u \in \mathrm{Hom}(a_m, a_n)$ by $|u|$, then one has the relation:

$$|u| = \mathrm{rank}\, u + n - m\,. \tag{4.3}$$

Relation (4.2) can also be written as:

$$\mathrm{Hom}(a_m, a_n) = \Omega(L, \mathrm{Hom}(E_m, E_n))[n - m]\,, \tag{4.4}$$

upon using standard mathematical notation for shifting degrees.

It is useful to consider the total boundary state space $\mathcal{H} = \oplus_{m,n}\mathrm{Hom}(a_m, a_n) = \Omega^*(L, \mathrm{End}(\mathbf{E}))$, where $\mathbf{E} = \oplus_n E_n$. This has an obvious $\mathbf{Z}\times\mathbf{Z}$ grading given by $\mathcal{H}_{m,n} = \mathrm{Hom}(a_m, a_n)$, and a $\mathbf{Z}_2$ grading given by:

$$\begin{aligned}\mathcal{H}_{\rm even} &= \oplus_{k+m-n=even}\mathrm{Hom}^k(a_m, a_n)\,,\\ \mathcal{H}_{\rm odd} &= \oplus_{k+m-n=\rm odd}\mathrm{Hom}^k(a_m, a_n)\,.\end{aligned} \tag{4.5}$$

The $\mathbf{Z}_2$ grading can be described in standard supergeometry language as follows. Let us define even and odd subbundles of $\mathbf{E}$ through:

$$E_{\rm even} = \oplus_{n=even}E_n\,, \qquad E_{\rm odd} = \oplus_{n=\rm odd}E_n\,. \tag{4.6}$$

Then $\mathbf{E} = E_{\rm even}\oplus E_{\rm odd}$ can be regarded as a super-vector bundle ($\mathbf{Z}_2$-graded bundle). Its bundle of endomorphisms then has the decomposition:

$$\begin{aligned}\mathrm{End}(\mathbf{E}) = {}& \mathrm{Hom}(E_{\rm even}, E_{\rm even}) \oplus \mathrm{Hom}(E_{\rm odd}, E_{\rm odd}) \oplus \mathrm{Hom}(E_{\rm even}, E_{\rm odd}) \oplus\\ &\oplus \mathrm{Hom}(E_{\rm odd}, E_{\rm even})\end{aligned} \tag{4.7}$$

and a total $\mathbf{Z}_2$ grading:

$$\begin{aligned}\mathrm{End}(\mathbf{E})_{\rm even} &= \mathrm{Hom}(E_{\rm even}, E_{\rm even}) \oplus \mathrm{Hom}(E_{\rm odd}, E_{\rm odd})\,,\\ \mathrm{End}(\mathbf{E})_{\rm odd} &= \mathrm{Hom}(E_{\rm even}, E_{\rm odd}) \oplus \mathrm{Hom}(E_{\rm odd}, E_{\rm even})\,.\end{aligned} \tag{4.8}$$

Then $\mathcal{H}$ corresponds to the the tensor product $\Omega^*(L)\otimes_{\Omega^0(L)}\Omega^0(\mathrm{End}(\mathbf{E}))$, in which case the $\mathbf{Z}_2$ grading on $\mathcal{H}$ is induced by the standard $\mathbf{Z}_2$ grading on $\Omega^*(L)$ and the $\mathbf{Z}_2$-grading (4.8) on $\mathrm{End}(\mathbf{E})$.

Also note that $\mathrm{End}(\mathbf{E})$ has a diagonal $\mathbf{Z}$-grading, whose mod 2 reduction gives the grading (4.7):

$$\mathrm{End}(\mathbf{E})_s = \oplus_{m-n=s}\mathrm{Hom}(E_m, E_n)\,. \tag{4.9}$$

As a consequence, the total boundary state space $\mathcal{H}$ has a $\mathbf{Z}$-grading induced by (4.9) and by the grading of $\Omega^*(L)$ through form rank. This coincides with the grading (4.3) given by the worldsheet U(1) charge, and whose mod 2 reduction is the string field theoretic grading (4.5).

(2) One has boundary products, which in our case are given (up to signs) by the wedge product of bundle valued forms (which are taken to involve composition of morphisms between the fibers):

$$u\cdot v = (-1)^{(k-n)\,\mathrm{rank}\,v}u\wedge v\,, \qquad \text{for } u\in\mathrm{Hom}(a_n, a_k)\,,\quad \text{and } v\in\mathrm{Hom}(a_m, a_n)\,. \tag{4.10}$$

This gives compositions of the form $\mathrm{Hom}(a_n, a_k)\times\mathrm{Hom}(a_m, a_n)\rightarrow\mathrm{Hom}(a_m, a_k)$. As in [12], these induce a total boundary product on $\mathcal{H}$ through:

$$u\cdot v = \oplus_{m,k}\left[\sum_n u_{nk}v_{mn}\right] \tag{4.11}$$

for elements $u = \oplus_{nk}u_{nk}$ and $v = \oplus_{mn}v_{mn}$ with $u_{nk}\in\mathrm{Hom}(a_n, a_k)$ and $v_{mn}\in\mathrm{Hom}(a_m, a_n)$.

The total boundary product also admits a standard supergeometric interpretation. Indeed, remember that both of the factors $\Omega^*(L)$ and $\Omega^0(\mathrm{End}(\mathbf{E}))$ admit natural structures of superalgebras, with multiplications given by wedge product of forms and composition of bundle morphisms, respectively. This allows us to consider the induced superalgebra structure on $\mathcal{H}$, which, following [18], corresponds to $\Omega^*(L)\hat{\otimes}_{\Omega^0(L)}\Omega^0(\mathrm{End}(\mathbf{E}))$. According to standard supermathematics, the corresponding product on $\mathcal{H}$ acts on decomposable elements $u = \omega \otimes f$ and $v = \eta \otimes g$ as follows:

$$(\omega \otimes f)(\eta \otimes g) = (-1)^{\pi(f)\,\mathrm{rank}\,\eta}(\omega \wedge \eta) \otimes (f \circ g)\,, \tag{4.12}$$

for ω, η some forms on L and $f, g \in \mathrm{End}(\mathbf{E})$. In this relation, $\pi(f)$ stands for the parity of f with respect to the decomposition (4.8).

If we apply this relation for $u \in \mathrm{Hom}(a_n, a_k)$ and $v \in \mathrm{Hom}(a_m, a_n)$, then $\pi(f)$ is the mod 2 reduction of $k - n$ and we recover relation (4.10) upon viewing u and v as bundle-valued forms. In local coordinates, one can write write $u = dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_r} u_{\alpha_1 \cdots \alpha_r}$, $v = dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_s} v_{\alpha_1 \cdots \alpha_s}$, and obtain:

$$uv = (-1)^{s(k-n)} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{r+s}} u_{\alpha_1 \cdots \alpha_r} v_{\alpha_{r+1} \cdots \alpha_{r+s}}\,, \tag{4.13}$$

which recovers the boundary products introduced above. This means that one has a sign factor of -1 each time one commutes dx^α with an odd bundle morphism. In our conventions, one writes the bundle morphisms to the right.

(3) One has bilinear forms ${}_{mn}\langle .,. \rangle_{nm}$ on the products $\mathrm{Hom}(a_m, a_n) \times \mathrm{Hom}(a_n, a_m)$, which are induced by the two-point boundary correlator on the disk. These can be identified through a localization argument and are given by:

$${}_{mn}\langle u, v \rangle_{nm} = \int_{L_n} \mathrm{tr}_{E_n}(u \cdot v) = (-1)^n \int_L \mathrm{tr}_{E_n}(u \cdot v)\,, \tag{4.14}$$

where tr_{E_n} denotes the fiberwise trace on $\mathrm{End}(E_n)$.

Since integration of forms requires the specification of an orientation, one must keep track of which of the two orientations of L is used in the definition of each bilinear form. This gives the crucial sign factor in the right-hand side. As in [12], one can combine these into a bilinear form $\langle .,. \rangle$ on the total boundary state space $\mathcal{H}$. Due to the sign factor in (4.14), one obtains:

$$\langle u, v \rangle = \int_L \mathrm{str}(uv)\,, \qquad \text{for } u, v \in \mathcal{H}\,, \tag{4.15}$$

where str is the supertrace with respect to the decomposition (4.8) (see [18] for details).

(4) One has degree one BRST operators $Q_{mn} : \mathrm{Hom}(a_m, a_n) \to \mathrm{Hom}(a_m, a_n)$, which in our case are given by the covariant differentials d_{mn} associated with the flat connections A_m and A_n. More precisely, d_{mn} is the differential of the "twisted" de

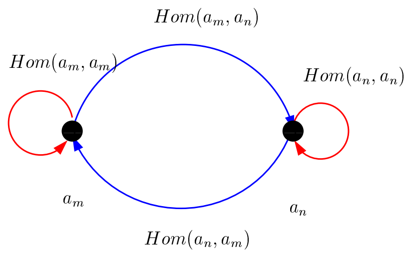

Figure 1: A full two-object subcategory of the category describing our D-brane system.

Rham complex:

$$
\begin{aligned}
0 &\xrightarrow{d_{mn}} \Omega^0(L, \mathrm{Hom}(E_m, E_n)) \xrightarrow{d_{mn}} \Omega^1(L, \mathrm{Hom}(E_m, E_n)) \xrightarrow{d_{mn}} \Omega^2(L, \mathrm{Hom}(E_m, E_n)) \\
&\xrightarrow{d_{mn}} \Omega^3(L, \mathrm{Hom}(E_m, E_n)) \xrightarrow{d_{mn}} 0\,.
\end{aligned}
$$

determined by the connection ∇_{mn} induced by A_m and A_n on $\mathrm{Hom}(E_m, E_n)$.

These can be combined into the total BRST operator $Q = \oplus_{m,n} Q_{mn}$, which can be identified as the exterior differential on $\mathbf{E}$ induced by the total connection $A = \oplus_n A_n$. This is a differential operator on $\Omega^*(\mathrm{End}(\mathbf{E}))$ with the property:

$$
Q(\omega \otimes f) = d\omega \otimes f + (-1)^{\mathrm{rank}\,\omega} \omega Q f\,, \tag{4.16}
$$

so it gives a superconnection on the superbundle $\mathbf{E}$ in the sense of [18].

It is not hard to see that all of the axioms discussed in [12] (and reviewed in section 2) are satisfied. In the language of section 2, we have a differential graded category $\tilde{\mathcal{A}}$ with objects a_n and bilinear forms between the morphism spaces which are invariant with respect to the BRST charges Q_{mn} and with respect to morphism compositions. The abstract structure is depicted in figure 1.

According to the axioms of section 2, the open string field theory is described by the action:

$$
S(\phi) = \frac{1}{2}\langle \phi, Q\phi \rangle + \frac{1}{3}\langle \phi, \phi \cdot \phi \rangle\,, \tag{4.17}
$$

where the string field $\phi = \oplus_{m,n} \phi_{mn}$ (with $\phi_{mn} \in \mathrm{Hom}(a_m, a_n)$) is a degree one element of the total boundary state space $\mathcal{H}$. Combining all of the data above, one can re-write this in the form:

$$
S(\phi) = \int_L \mathrm{str}\left[\frac{1}{2}\phi Q \phi + \frac{1}{3}\phi\phi\phi\right]. \tag{4.18}
$$

The product in the integrand is given by (4.10).

5. Complex conjugations

Our string field theory can be endowed with conjugations, provided that the collection of D-branes $(\mathbf{L}_n, E_n, A_n)$ is invariant with respect to the operation which takes

JHEP06(2001)064

a brane into a conjugate brane.[9] We propose that the conjugate brane of a D-brane $a = (\mathbf{L}, E, A)$ is the D-brane $\overline{a} = (\overline{\mathbf{L}}, \overline{E}, A^*)$ defined as follows:

(a) The underlying cycle of $\overline{a}$ is L.

(b) If the grading of a is given by n (i.e. $\mathbf{L} = (L, n)$), then the grading of $\overline{a}$ is $\overline{n} = -n$.

(c) The underlying bundle $\overline{E}$ is the antidual of the bundle E, i.e. the bundle whose fiber E_x at a point x of L is the space of *antilinear* functionals defined on the fiber E_x of E.

(d) The connection $\overline{\nabla}$ is given by:

$$(\overline{\nabla}_X \psi)(s) = -\psi(\nabla_{\overline{X}}(s)) + X\left[\psi(s)\right], \tag{5.1}$$

for any local sections s of E and ψ of $\overline{E}$ and any vector fields X. In the right-hand side, $\overline{X}$ is the complex conjugate of X. Namely, viewing X as a complex-linear derivation of the algebra of (complex-valued) functions on L, one takes:

$$\overline{X}(f) := \overline{X(\overline{f})}\,. \tag{5.2}$$

The result is another complex-linear derivation, i.e. a vector field. If x^α are (real !) local coordinates on L, then upon expanding $X = X^\alpha \partial_\alpha$, one has $\overline{X} = \overline{X^\alpha}\partial_\alpha$, where $\overline{X^\alpha}$ denotes the usual complex conjugate of the function X^α. In particular, $\overline{\partial}_\alpha = \partial_\alpha$.

If s_α is a local frame of sections for E, then we can consider the antidual frame $\overline{s}_\alpha$ of $\overline{E}$, defined by the conditions:

$$\overline{s}_\alpha(s_\beta) = \delta_{\alpha\beta}\,. \tag{5.3}$$

Then we have (1-form valued) connection matrices A and A^* defined through:

$$\begin{aligned} \nabla(s_\alpha) &= A_{\beta\alpha} s_\beta\,, \\ \overline{\nabla}(\overline{s}_\alpha) &= A^*_{\beta\alpha} \overline{s}_\beta\,. \end{aligned} \tag{5.4}$$

In this case, we obtain the relation:

$$A^* = -A^+\,, \tag{5.5}$$

i.e.

$$A^*_i(x) = -A_i(x)^+\,, \tag{5.6}$$

where $A = A_i(x)dx^i$ and $A^* = A^*_i(x)dx^i$.

[9] Our conjugate branes should *not* be identified with the "topological antibranes" of [3, 5]. The "topological antibranes" of those papers result from our conjugate branes upon performing a further shift of the grading by 1. The conjugation operators we construct below are related with the "gauge invariance" of [3, 5], which consists in shifting the grading of all D-branes by 1 combined with reversing the role of branes and antibranes.

The abstract description given above has the advantage that it does not require the choice of supplementary data. It is possible to formulate a conjugation operator in these abstract terms, and check all relevant axioms given in section 2. Below, I shall use a mathematically less elegant, but more concrete approach which requires the choice of a metric on the bundle E.[10] For this, let us assume that the bundle E is endowed with a hermitean metric h_E; the precise choice of such a metric is irrelevant for what follows. We use standard *physics* conventions by taking h_E to be antilinear in its first variable and linear in the second. Such a scalar product defines a *linear* isomorphism between $\overline{E}$ and E, which identifies an antilinear functional ψ on E_x with the vector $u_\psi \in E_x$ satisfying the equation:

$$h_E(v, u_\psi) = \psi(v)\,, \qquad \text{for all } v \in E_x\,. \tag{5.7}$$

Using this isomorphism, one can translate the abstract definition of "conjugate branes" into the following more concrete description:

(c') The underlying bundle $\overline{E}$ of $\overline{a}$ can be identified with the underlying bundle E of A

(d') The background flat connection A^* of E can be identified with the opposite of the hermitean conjugate of A with respect to the metric h_E:

$$A^* = -A^+\,. \tag{5.8}$$

From now on, the symbol A^+ always denotes hermitean conjugation with respect to h_E, unless we state otherwise. We can now describe the antilinear conjugations of our string field theory. Since this is a bit subtle, we shall proceed in two steps.

Step 1. Given a (complex-valued) differential form $\omega = \omega_{\alpha_1 \cdots \alpha_k} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$ on L, we define its conjugate by:[11]

$$\tilde{\omega} := \overline{\omega_{\alpha_1 \cdots \alpha_k}} dx^{\alpha_k} \wedge \cdots \wedge dx^{\alpha_1} = (-1)^{k(k-1)/2} \overline{\omega}\,, \tag{5.9}$$

where $\overline{\omega} = \overline{\omega_{\alpha_1 \cdots \alpha_k}} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$ is the usual complex conjugate of ω and we used the relation:

$$dx^{\alpha_k} \wedge \cdots \wedge dx^{\alpha_1} = (-1)^{k(k-1)/2} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}\,. \tag{5.10}$$

It is not hard to check that this operation has the following properties:

$$\begin{aligned} (\alpha \wedge \beta)^{\tilde{}} &= \tilde{\beta} \wedge \tilde{\alpha}\,, \\ d(\tilde{\omega}) &= (-1)^{\mathrm{rank}\,\omega} (d\omega)^{\tilde{}}\,, \\ \int_L \tilde{\omega} &= -\overline{\int_L \omega}\,. \end{aligned} \tag{5.11}$$

[10]This is not natural in a topological string theory, which should be formulated as much as possible without reference to a metric. I prefer to use the metric language in order to make the paper easier to understand for the casual reader.

[11]x^α are *real* coordinates on the three-cycle L (which is not a complex manifold !). I hope this avoids any confusion.

Step 2. Let us now return to our our collection of D-branes $a_n = (\mathbf{L}_n, E_n, A_n)$ and assume that we have picked hermitean metrics $h_n = h_{E_n}$ on each of the bundles E_n. For $u = \omega \otimes f$ a decomposable element of $\mathrm{Hom}(a_m, a_n)$ (with ω a complex-valued form on L and $f \in \mathrm{Hom}(E_m, E_n)$), we define:

$$*u := (-1)^{(n-m+1)rk\omega} \tilde{\omega} \otimes f^+ . \tag{5.12}$$

We then extend this uniquely to an antilinear operation from $\Omega^*(L, \mathrm{Hom}(E_m, E_n))$ to $\Omega^*(L, \mathrm{Hom}(E_n, E_m))$. In terms of the usual hermitean conjugation of bundle-valued forms ($(\omega \otimes f)^+ := \overline{\omega} \otimes f^+$), this reads:

$$*u = (-1)^{rku(rku+1)/2+(n-m)rku} u^+ . \tag{5.13}$$

Since the choice of hermitean metrics allows us to identify the "conjugate branes" $\overline{a}_n = (\overline{L_n}, \overline{E_n}, A_n^*)$ with the triples $(\mathbf{L}_{-n}, E_n, -A_n^+)$, we can also view (5.12) as antilinear maps $*_{mn}$ from $\mathrm{Hom}(a_m, a_n)$ to $\mathrm{Hom}(\overline{a_n}, \overline{a_m})$. We claim that these operators give conjugations of our string field theory. To see this, one must check that the axioms of section 2 are satisfied. For this, notice first that the operations $*_{mn}$ are homogeneous of degree zero when viewed as applications between $\mathrm{Hom}(a_m, a_n)$ and $\mathrm{Hom}(\overline{a_n}, \overline{a_m})$. This follows from the fact that they preserve form rank, and from form our definition $\overline{n} = -n$, $\overline{m} = -m$ of gradings for the "conjugate branes":

$$| *_{mn} u_{mn}| = \mathrm{rank}(*_{mn} u_{mn}) + (-m) - (-n) = \mathrm{rank}\, u_{mn} + n - m = |u_{mn}| . \tag{5.14}$$

The other properties listed in subsection 2.2. then follow by straightforward computation, and I shall leave their verification as an exercise for the reader.

It should be clear from our discussion that the theory considered in the previous section will be invariant under conjugation only if the set of D-branes a_n is invariant with respect to the involution $a_n \rightarrow \overline{a_n}$. This can always be achieved by adding the "conjugate branes" $\overline{a_n}$ to the original set, but in general this will lead to more than one D-brane wrapped on L for each grade n. Hence a general analysis requires that we allow various D-branes present in the system to have the same grade n. While it is possible to carry out our analysis for such general systems, this leads to rather complicated notation. When discussing reality issues in this paper, we shall assume for simplicity that the set of D-branes present in our background is invariant with respect to the transformation $a_n \rightarrow \overline{a_n}$. This requires that the set of grades under consideration is invariant with respect to the substitution $n \rightarrow -n$ and that $E_{-n} = E_n$, $h_{E_{-n}} = h_{E_n}$ and $A_{-n} = -A_n^+ = A_n$. With these hypotheses, one has $\overline{a}_n = a_{-n}$, and we can impose the reality constraint $*\phi_{mn} = \phi_{-n,-m}$ on the string field. Note, however, that there is no reason why our background should contain only brane-"conjugate brane" pairs. If this condition is not satisfied, then one can simply work with a *complex* action of the form (2.8). In fact, complex string field actions are natural in topological string field theories, as is well known from the example of the B-model. A real string field action is only required in a physically complete theory, such as the theory of "all" topological D-branes.

JHEP06(2001)064

6. An extended action

Our theory (4.18) can be extended in the following manner. Consider the supermanifold $\mathcal{L} := \Pi TL$ obtained by applying parity reversal on the fibers of the tangent bundle of L. $\mathcal{L}$ is equipped with a sheaf $\mathcal{O}_\mathcal{L}$ of Grassmann algebras, whose sections are superfunctions defined on $\mathcal{L}$.

If θ^α ($\alpha = 1, \ldots, 3$) are odd coordinates along the fibers T_xL, then sections of $\mathcal{O}_\mathcal{L}$ are superfields of the form:

$$f(x,\theta) = f^{(0)}(x) + \theta^\alpha f^{(1)}_\alpha(x) + \theta^\alpha\theta^\beta f^{(2)}_{\alpha\beta}(x) + \theta^\alpha\theta^\beta\theta^\gamma f^{(3)}_{\alpha\beta\gamma}(x)\,, \tag{6.1}$$

with Grassmann-valued coefficients $f^{(k)}_{\alpha_1\cdots\alpha_k}$ (sections of $\hat{\mathcal{O}}_L = \mathcal{O}_L \otimes G$, where G is an underlying Grassmann algebra). The space $\Gamma(\mathcal{O}_\mathcal{L})$ of such superfields forms a $\mathbf{Z} \times \mathbf{Z}_2$-graded algebra, with $\mathbf{Z}$-grading[12] induced by the degree of the monomials in θ^α and $\mathbf{Z}_2$-grading given by Grassmann parity.

We are interested in elements $\mathbf{\Phi}$ of the space $\Gamma(\mathcal{O}_\mathcal{L}\hat{\otimes}\,\mathrm{End}(\mathbf{E}))$, which carries the $\mathbf{Z}_2$ grading induced from the two components of the tensor product. We shall denote this total $\mathbf{Z}_2$-grading by deg. Such bundle-valued superfields have the expansion:

$$\mathbf{\Phi}(x,\theta) = \sum_{k=0}^{3} \theta^{\alpha_1}\cdots\theta^{\alpha_k}\mathbf{\Phi}^{(k)}_{\alpha_1\cdots\alpha_k}(x)\,, \tag{6.2}$$

whose coefficients $\mathbf{\Phi}^{(k)}_{\alpha_1\cdots\alpha_k}$ are sections of the sheaf $\mathrm{End}(\mathbf{E}) := \hat{\mathcal{O}}_L \otimes \mathrm{End}(\mathbf{E})$, i.e. elements of the algebra $\Gamma(\mathrm{End}(\mathbf{E})) := G\hat{\otimes}_\mathbf{C}\Gamma(L, \mathrm{End}(\mathbf{E}))$ of Grassmann-valued sections of $\mathrm{End}(\mathbf{E})$. This algebra has a $\mathbf{Z}$-grading induced by the $\mathbf{Z}$-grading on $\mathrm{End}(\mathbf{E})$ and a total $\mathbf{Z}_2$-grading induced by the sum of Grassmann parity with the $\mathbf{Z}_2$-degree on $\mathrm{End}(\mathbf{E})$. Note that the $\bmod 2$ reduction of the $\mathbf{Z}$-grading does not give the $\mathbf{Z}_2$ grading.

The coefficients $\mathbf{\Phi}^{(k)}_{\alpha_1\cdots\alpha_k}$ define *Grassmann-valued* forms $\hat{\phi}^{(k)} = dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_k}\mathbf{\Phi}^{(k)}_{\alpha_1\cdots\alpha_k}(x)$, viewed as elements of the algebra $\mathcal{H}_e = \Omega^*(L)\hat{\otimes}\Gamma(\mathrm{End}(\mathbf{E}))$ of forms with Grassmann-valued coefficients in $\mathrm{End}(\mathbf{E})$. This algebra also has total $\mathbf{Z}$ and $\mathbf{Z}_2$-gradings (denoted again by $|.|$ and deg) which are induced by the $\mathbf{Z}$ and $\mathbf{Z}_2$-gradings on $\Omega^*(L)$ and $\Gamma(\mathrm{End}(\mathbf{E}))$. The $\mathbf{Z}_2$-grading deg is the $\bmod 2$ reduction of the sum between the $\mathbf{Z}$-grading $|.|$ and Grassmann parity. If $\hat{a}$ and $\hat{b}$ are elements of $\mathcal{H}_e$, then their product is:

$$\hat{a}\hat{b} = (-1)^{(\deg\hat{a} - \mathrm{rank}\,\hat{a})\,\mathrm{rank}\,\hat{b}}\,\hat{a}\wedge\hat{b}\,. \tag{6.3}$$

If $\deg(\mathbf{\Phi}) = p \in \mathbf{Z}_2$ is the parity of the superfield $\mathbf{\Phi}$, then the components $\mathbf{\Phi}^{(k)}_{\alpha_1\cdots\alpha_k}$ have total parity $\deg\mathbf{\Phi}^{(k)}_{\alpha_1\cdots\alpha_k} = p - k \pmod 2$, while the Grassmann-valued form $\hat{\phi}^{(k)}$ has total parity p. We obtain a correspondence $\mathbf{\Phi} \leftrightarrow \hat{\phi}$ which takes $\mathbf{\Phi}$ into the sum of Grassmann-valued forms $\hat{\phi} = \sum_{k=0}^{3}\phi^{(k)}$. This map is homogeneous of degree zero, i.e. it intertwines the $\mathbf{Z}_2$ gradings deg on $\Gamma(\mathcal{O}_\mathcal{L}\otimes\mathrm{End}(\mathbf{E}))$ and $\mathcal{H}_e$.

[12] The components of this grading are non zero only in degrees 0, 1, 2 and 3.

Under this correspondence, the BRST differential $Q = d$ on $\mathcal{H}_e$ (extended to Grassmann-valued forms in the obvious manner) maps to the operator:

$$D = \theta^\alpha \frac{\partial}{\partial x^\alpha} \tag{6.4}$$

on superfields. By analogy with the usual Chern-Simons case, this allows us to write an extended string field action:

$$S_e(\mathbf{\Phi}) = \int_L d^3x \int d^3\theta \, \mathrm{str}\left[\frac{1}{2}\mathbf{\Phi}\mathbf{D}\mathbf{\Phi} + \frac{1}{3}\mathbf{\Phi}\mathbf{\Phi}\mathbf{\Phi}\right]. \tag{6.5}$$

This can be viewed as a $\mathbf{Z}$-graded version of *extended* super-Chern-Simons field theory. It should be compared with the proposal of [20].

Observation. It is a standard subtlety (see, for example, [24]) that the product of the Grassmann-valued forms $\hat{\phi}^{(k)}$ induced by superfield multiplication differs from the standard wedge product. This is due to the fact that the coordinates θ^α are Grassmann odd. To be precise, let us consider two superfields $\mathbf{A}$, $\mathbf{B}$ of parities $p_\mathbf{A}$ and $p_\mathbf{B}$, and let $\mathbf{A}^{(k)}_{\alpha_1 \cdots \alpha_k}$ and $\mathbf{B}^{(l)}_{\alpha_1 \cdots \alpha_l}$ be their components under the expansion (6.1), which have total parities $p_\mathbf{A} - k \pmod 2$ and $p_\mathbf{B} - l \pmod 2$. We also consider the associated Grassmann-valued forms $\hat{a}^{(k)} = dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k} \mathbf{A}^{(k)}_{\alpha_1 \cdots \alpha_k}$ and $\hat{b}^{(l)} = dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_l} \mathbf{B}^{(l)}_{\alpha_1 \cdots \alpha_l}$. Then one can write:

$$(\mathbf{AB})(x, \theta) = \sum_{n=0}^{3} \theta^{\alpha_1} \cdots \theta^{\alpha_n} (\mathbf{AB})^{(n)}_{\alpha_1 \cdots \alpha_n}(x). \tag{6.6}$$

If $(\hat{a} * \hat{b})^{(n)} = dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_n} (\mathbf{AB})^{(n)}_{\alpha_1 \cdots \alpha_n}(x)$ is the associated Grassmann-valued form, then one has $(\hat{a} * \hat{b})^{(n)} = \sum_{k+l=n} \hat{a}^{(k)} * \hat{b}^{(l)}$, where:

$$\hat{a}^{(k)} * \hat{b}^{(l)} = (-1)^{(p_\mathbf{A} - k)l} \hat{a}^{(k)} \wedge \hat{b}^{(l)}. \tag{6.7}$$

The sign prefactor arises when commuting the odd variables θ^{β_j} with the coefficient $\mathbf{A}^{(k)}_{\alpha_1 \cdots \alpha_k}$ in the following relation:

$$\begin{aligned} \theta^{\alpha_1} \cdots \theta^{\alpha_k} \mathbf{A}^{(k)}_{\alpha_1 \cdots \alpha_k} \theta^{\beta_1} \cdots \theta^{\beta_l} \mathbf{B}^{(l)}_{\beta_1 \cdots \beta_l} &= (-1)^{(p_\mathbf{A} - k)l} \theta^{\alpha_1} \cdots \theta^{\alpha_k} \theta^{\beta_1} \cdots \theta^{\beta_l} \times \\ &\quad \times \mathbf{A}^{(k)}_{\alpha_1 \cdots \alpha_k} \mathbf{B}^{(l)}_{\beta_1 \cdots \beta_l} \end{aligned} \tag{6.8}$$

Equation (6.7) reads:

$$\hat{a} * \hat{b} = (-1)^{(p_\mathbf{A} - \mathrm{rank}\, \hat{a}) \, \mathrm{rank}\, \hat{b}} \hat{a} \wedge \hat{b} = (-1)^{(p_\mathbf{A} - |\mathbf{a}|) \, \mathrm{rank}\, \hat{b}} \hat{a} \cdot \hat{b}, \tag{6.9}$$

where the multiplication $\cdot$ is that of (4.10). Since $p_\mathbf{A} = \deg \hat{a}$, this is exactly the product (6.3) on the algebra $\mathcal{H}_e$. Hence one can write the extended action as a functional on $\mathcal{H}_e$:

$$S_e(\hat{\phi}) = \int_L d^3x \, \mathrm{str}\left[\frac{1}{2}\hat{\phi} * d\hat{\phi} + \frac{1}{3}\hat{\phi} * \hat{\phi} * \hat{\phi}\right]. \tag{6.10}$$

In this form, the action S_e can be related to the general extension procedure discussed in [33] (when the latter is translated in our conventions). One imposes the condition that $\mathbf{\Phi}$ is an odd superfield, i.e. $\deg \hat{\phi} = \hat{1} \in \mathbf{Z}_2$.

If $p_{\mathbf{A}} = \hat{1}$, then the product (6.9) agrees with the multiplication (4.10) when $|\mathbf{a}| = 1$. This implies that the extended action (6.5) reduces to the unextended one (4.17) if all non-vanishing Grassmann-valued form components of the extended string field $\mathbf{\Phi}$ satisfy $|\hat{\phi}_{mn}| = 1$.

7. The moduli space of vacua

Recall that the moduli space of vacua is built by solving the Maurer-Cartan equation:

$$Q\phi + \frac{1}{2}[\phi,\phi] = 0 \Longleftrightarrow Q\phi + \phi\phi = 0 \tag{7.1}$$

for string fields $\phi \in \mathcal{H}$ of degree $|\phi| = 1$. Two solutions of (7.1) are identified if they differ by a gauge transformation. The gauge group is generated by infinitesimal transformations of the form $\phi \to \phi + Qu + [\phi, u]$. If our collection of D-branes is invariant under complex conjugation, one can also impose the reality constraint $*\phi_{m,n} = \phi_{-n,-m}$, in which case one also requires reality ($*u = u$) of the gauge generator u. The commutator appearing in (7.1) is the graded commutator on the algebra $\mathcal{H}$, which reduces to the square of ϕ due to the degreee one condition $|\phi| = 1$. The resulting moduli space is a (formal) supermanifold $\mathcal{M}$, which can be built locally by solving (7.1) upon expressing ϕ as a formal power series in appropriate variables [23].

The degree one condition on the string field reads:

$$\operatorname{rank}\phi_{mn} + n - m = 1 \Longleftrightarrow \operatorname{rank}\phi_{mn} = 1 + m - n\,. \tag{7.2}$$

As in [12], this implies that one can deform the vacuum by condensing fields of rank $1 + m - n$ in the boundary sector $\mathrm{Hom}(a_m, a_n)$. The fact that one can condense higher rank forms in this manner is due to the presence of $\mathbf{Z}$-graded branes in the background. In fact, it can be argued that the standard deformations (7.1) represent *extended* deformations of the usual vacuum based on the collection of flat connections A_n.

7.1 The shift-invariant case

To understand this, let us consider the case $E_m = E$ and $A_m = A$ for all m, i.e. we take the underlying bundles and flat connections to be identical. We also take an infinity of D-branes, i.e. we let n run over all integer values from $-\infty$ to $+\infty$. In this case, the moduli space built by solving (7.1) will in fact be ill-defined, due to the existence of a countable number of shift symmetries $S_k : \phi_{m,n} \to \phi_{m+k,n+k}$ of the string field action.

This problem is easily solved by restricting to "shift-invariant" configurations, i.e. configurations of the string field for which $\phi_{mn} = \phi_{m-n}$ depends only on the difference $m - n$.[13] We thus replace $\mathcal{M}$ by the moduli space $\mathcal{M}_d$, which is obtained by solving the Maurer-Cartan eqs. (7.1) for shift invariant configurations. For such configurations, the equations reduce to:

$$Q_n \phi_n + \sum_{k+l=n} \phi_k \phi_l = 0\,, \tag{7.3}$$

where $Q_n := d_A$ acts on $\mathrm{Hom}(a_n, a_0) = \Omega^*(L, \mathrm{End}(E))[-n]$ and ϕ_n is a degree one element of $\mathrm{Hom}(a_n, a_0)$, i.e. a form of rank $1 + n$ valued in $\mathrm{End}(E)$.

We next consider the linearization $Q_n \phi_n = 0$ of (7.3), which describes infinitesimal vacuum deformations. Upon dividing through linearized gauge transformations ($\phi_n \to \phi_n + Q_n u_n$), we obtain that the tangent space to $\mathcal{M}_d$ (at the point $O = \{A_n = A \text{ for all } n\}$) is given by the degree one BRST cohomology of $\mathcal{H}_d := \oplus_n \Omega^*(L, \mathrm{End}(E))[-n]$

$$T_O \mathcal{M}_d = H^1_Q(\mathcal{H}_d) = \frac{\ker\,[Q : \mathcal{H}^1_d \to \mathcal{H}^2_d]}{\mathrm{im}\,[Q : \mathcal{H}^0_d \to \mathcal{H}^1_d]}\,. \tag{7.4}$$

It is easy to see that this coincides with the *total* BRST cohomology of $\Omega^*(L, \mathrm{End}(E))$:

$$T_O(\mathcal{M}_d) = H^*_{d_A}(L, \mathrm{End}(E)) = \oplus_{k=0}^3 H^k_{d_A}(L, \mathrm{End}(E))\,. \tag{7.5}$$

That is, (shift-invariant) *degree one deformations of our theory correspond to extended deformations of the moduli space of flat connections on* E. It is well known (see, for example, [26]) that the moduli space of flat connections on E is the same as the moduli space of (classical) vacua of the Chern-Simons theory based on E, which in turn gives the (large radius) description of the boundary sector of a *single* D-brane $(\mathbf{L}_0, E, A)$ wrapped on L. It follows that inclusion of graded D-branes leads automatically to an *extended* moduli space. This is a particular realization of the general principle (already mentioned in [13]) that usual (degree one) deformations of a shift-completed string field theory describe *extended* deformations of the uncompleted system.

Let us compare this with a theory of non-graded branes. If one were to neglect the fact that type A branes are $\mathbf{Z}$-graded, and consider the naive identification of D-branes with the data (L, E, A), then, as in [26], one would arrive at the conclusion that the string field theory of the D-brane collection $(a_n)_n$ is the standard Chern-Simons field theory based on the bundle $E_{\mathrm{tot}} = E_n^{\oplus \mathbf{Z}}$ (viewed as an *even* bundle rather than a superbundle) and considered around the vacuum described by the con-

[13] A more careful treatment requires an analysis of shift-equivariant string field configurations, i.e. configurations which are shift invariant only up to gauge transformations. Here we give a simplified discussion in terms of shift-invariant solutions.

figuration $\{A_n = A \text{ for all } n\}$. Vacuum deformations of this theory would correspond to independent deformations $\phi_{n,n}$ of the flat connections A_n, and would locally give an infinite power $\mathcal{M} = \mathcal{M}_0^{\mathbf{Z}}$ of the moduli space $\mathcal{M}_0$ of a single flat connection A. Restricting to shift-invariant deformations would then give the "regularized" moduli space $\mathcal{M}_d = \mathcal{M}_0$. Tangent direction to the latter are described by bundle-valued *one*-forms $\omega \in \Omega^1(L, \mathrm{End}(E))$.

7.2 Degree constraints

Let us now return to the general case of distinct bundles E_n. We would like to use constraint (7.2) in order to understand the types of D-brane configurations which result from physical vacuum deformations.

For this, note that presence of the D-branes a_m and a_n in the string background requires consideration of both ϕ_{mn} and ϕ_{nm} as components of the string field. Since one requires that the total string field have degree one, then one obtains constraints in the case $\phi_{nm} \neq 0$ or $\phi_{mn} \neq 0$, due to the fact that the ranks of these components must belong to the set $\{0, 1, 2, 3\}$. In fact, *both* of these components will vanish unless $|m - n| \leq 2$, i.e. $m - n \in \{-2, -1, 0, 1, 2\}$. Since (7.2) simply requires rank $\phi_{mm} = 1$ for $m = n$, we have *six* possibilities for which at least one of ϕ_{mn} and ϕ_{nm} can be non zero:

(1) $|m - n| \leq 2$, which gives 5 possibilities, further subdivided as follows:

(a) $|m - n| = 2$, i.e. $n \in \{m - 2, m + 2\}$, in which case only one of ϕ_{mn} and ϕ_{nm} can be non zero

(b) $|m - n| = 1$, i.e. $n \in \{m - 1, m + 1\}$, in which case both ϕ_{mn} and ϕ_{nm} can be non zero.

(c) $m = n$, in which case $\phi_{mn} = \phi_{nm} = \phi_{mm}$ can be non zero.

(2) $|m - n| > 2$, which requires $\phi_{mn} = \phi_{nm} = 0$.

This can be visualized in the following manner. Let us identify the D-branes a_n as abstract points of a regular one-dimensional lattice, and view the components ϕ_{mn} as link variables connecting the various nodes, with ϕ_{mm} viewed as self-linking of a node with itself. The links ϕ_{mn} for $m \neq n$ are oriented (since ϕ_{mn} and ϕ_{nm} should be viewed as independent data), while the self-links ϕ_{mm} carry no orientation.

Then (1) and (2) tell us that two nodes m and n can be connected by a link if and only if $|m - n| \leq 2$. Physically, this means that condensation of string field components is *local* with respect to the grade n, i.e. our system behaves in certain ways like a lattice with finite length interactions. This observation gives a string field theoretic interpretation of the point made in [3] that the grade should in a certain sense be "$\mathbf{Z}_6$-valued". In the conjugation-invariant case, the reality condition $*\phi_{mn} = \phi_{-n,-m}$ puts further constraints on the allowed configurations, without modifying the qualitative picture discussed above.

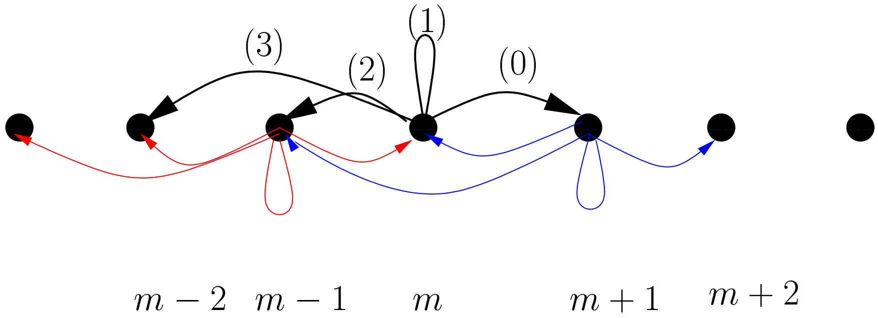

Figure 2: The formal lattice describing the allowed string field configurations. The numbers in round brackets indicate form degree.

8. Analysis of deformations

Consider the general expansion of a degree one string field:

$$\phi = \oplus_{mn}\phi_{mn} \, , \tag{8.1}$$

with $\phi_{mn} \in \mathrm{Hom}^1(a_m, a_n) = \Omega^{1+m-n}(L, \mathrm{Hom}(E_m, E_n))$.

Only terms with $n - m = -2, -1, 0$ or 1 survive, so we obtain:

$$\phi = \oplus_m \left[\phi^{(3)}_{m,m-2} \oplus \phi^{(2)}_{m,m-1} \oplus \phi^{(1)}_{m,m} \oplus \phi^{(0)}_{m,m+1}\right] , \tag{8.2}$$

where the superscripts indicate the rank of forms. The Maurer-Cartan equations read:

$$d\phi^{(1+m-k)}_{mk} + \sum_n \phi^{(1+n-k)}_{nk}\phi^{(1+m-n)}_{mn} = 0 \, . \tag{8.3}$$

Considering the non-trivial cases $k = m-1, m, m+1, m+2$, this gives:

$$\begin{aligned} d\phi^{(2)}_{m,m-1} + \phi^{(0)}_{m-2,m-1}\phi^{(3)}_{m,m-2} + \phi^{(1)}_{m-1,m-1}\phi^{(2)}_{m,m-1} + \phi^{(2)}_{m,m-1}\phi^{(1)}_{m,m} + &\\ + \phi^{(3)}_{m+1,m-1}\phi^{(0)}_{m,m+1} &= 0 \, , \\ d\phi^{(1)}_{m,m} + \phi^{(0)}_{m-1,m}\phi^{(2)}_{m,m-1} + \phi^{(1)}_{m,m}\phi^{(1)}_{m,m} + \phi^{(2)}_{m+1,m}\phi^{(0)}_{m,m+1} &= 0 \, , \\ d\phi^{(0)}_{m,m+1} + \phi^{(0)}_{m,m+1}\phi^{(1)}_{m,m} + \phi^{(1)}_{m+1,m+1}\phi^{(0)}_{m,m+1} &= 0 \, , \\ \phi^{(0)}_{m+1,m+2}\phi^{(0)}_{m,m+1} &= 0 \, , \end{aligned} \tag{8.4}$$

We obtain 4 systems of equations Σ_k ($k = 1, \ldots, 4$), where Σ_k correspond to the k-th row in (8.4). Each system contains a number of equations parameterized by m. In a theory with conjugations, each of the systems Σ_k is independently conjugation invariant, as a consequence of the reality condition $*\phi_{mn} = \phi_{-n,-m}$ on the string field. These equations are a particular realization of similar constraints holding in an arbitrary cubic string field theory with D-branes, a more systematic study of which will be given in [14]. Here I shall only make a few basic observations.

8.1 Diagonal deformations

Let us consider the particular case $\phi_{mn} = \delta_{mn}\phi^{(1)}_{mm}$, which corresponds to condensing boundary operators in each diagonal sector, but no boundary condition changing operators. In this situation, eqs. (8.4) reduce to:

$$d\phi^{(1)}_{mm} + \phi^{(1)}_{mm}\phi^{(1)}_{mm} = 0\,, \tag{8.5}$$

with $d = d_{mm}$, the differential on $\Omega^*(L, \mathrm{End}(E_m))$ induced by the flat connection A_m. Since $\phi^{(1)}_{mm}$ are one-forms valued in $\mathrm{End}(E_m)$, and the product (4.10) reduces in this case to the usual wedge product of forms, this can also be written as:

$$d\phi^{(1)}_{mm} + \phi^{(1)}_{mm} \wedge \phi^{(1)}_{mm} = 0\,. \tag{8.6}$$

These are the standard equations describing *independent* deformations $A_m \to A'_m := A_m + {\phi^{(1)}}_{mm}$ of the flat connections A_m. This is a realization of the general principle [12, 13] that condensation of diagonal components of the string field can be interpreted as performing independent deformations of the D-branes a_n. Upon integrating such deformations, we obtain a D-brane system of the type $(a'_n)_n$, where a'_n is a deformation of the brane a_n, obtained by modifying its flat background connection. More precisely, one obtains $a'_n = (\mathbf{L}_n, E_n, A'_n)$, with the new background connection A'_n. In the language of [12, 13], such deformations preserve the category structure and thus they do not lead to D-brane composite formation.

8.2 Unidirectional off-diagonal deformations and exotic type A branes

Let us now consider the case where all ϕ_{mm-2}, ϕ_{mm-1} and ϕ_{mm} vanish vanish but ϕ_{mm+1} may be non zero. This corresponds to condensing degree zero forms $\phi^{(0)}_{mm+1}$ in each boundary condition changing sector $\mathrm{Hom}(a_m, a_{m+1})$. Note that $\phi^{(0)}_{mm+1}$ is simply a bundle morphism from E_m to E_{m+1}.

In this case, the constraints (8.4) reduce to:

$$\begin{aligned} \phi^{(0)}_{m+1,m+2}\phi^{(0)}_{m,m+1} &= 0\,, \\ d\phi^{(0)}_{m,m+1} &= 0\,. \end{aligned} \tag{8.7}$$

The second condition tells us that the morphism ϕ_{mm+1} is flat (covariantly constant) with respect to the connection induced by A_m and A_{m+1} on the bundle $\mathrm{Hom}(E_m, E_{m+1})$, while the first equation shows that these morphisms form a *complex*:

$$\cdots E_{m-1} \xrightarrow{\phi_{m-1,m}} E_m \xrightarrow{\phi_{m,m+1}} E_{m+1} \cdots\,. \tag{8.8}$$

As explained on general grounds in [12, 13], condensation of such operators destroys the original category structure, leading to a so-called *collapsed category.* If we restrict to the simplest case when the collection of non-zero condensates ϕ_{mm+1}

connects together all of the branes present in our system,[14] then the end result of such a condensation process is a single D-brane composite. It follows that one can produce an *entirely new object* associated with the cycle L through such a process. Such an object corresponds to a complex of the type (8.8), whose morphisms are covariantly constant with respect to the original connections A_n. Complexes of this type were studied from a mathematical perspective in [19]. We wish to stress that the entire complex (8.8) must be viewed as a *new* topological D-brane (a D-brane composite) in this situation. In the case when some of the morphisms $\phi_{m,m+1}$ vanish, the complex (8.8) splits into connected subcomplexes, and each subcomplex should be viewed as a novel D-brane composite.

We conclude that, *given a special lagrangian cycle L, one has many more topological D-branes wrapping L than the graded branes of the type* $(\mathbf{L}_n, E_n, A_n)$. This already gives a vast enlargement of the category of topological D-branes, as was already mentioned in [13]. Of course, an even bigger enlargement can be obtained by considering more than one lagrangian cycle, for example by taking a collection of lagrangians with transverse intersections. This will be discussed in detail somewhere else.

8.3 General deformations

It is clear that a generic deformation satisfying (8.4) does not correspond to a flat connection background on the cycle L; indeed, such deformations involve condensation of forms of rank *zero*, *two* and *three*, beyond the standard condensation of rank one forms. The resulting string field backgrounds therefore lead to quite exotic classes of topological D-brane composites. The formalism appropriate for studying such systems is the theory of flat superconnections on $\mathbf{Z}$-graded superbundles, the basics of which were developed by Bismutt and Lott in [19]. This must be combined with the foundational work of [12, 13] and with the mathematical discussion of Bondal and Kapranov [17].

8.3.1 General string field backgrounds and pseudocomplexes

While I will not give a complete analysis along these lines, I wish to explain the relation between general solutions to (8.4) and the framework of [12, 13]. For this, let us once again restrict to the shift-invariant case $E_n = E$ and $A_n = A$ for all n, and let us assume that n runs over all signed integers.

In the language of [13], our category $\tilde{\mathcal{A}}$ built on the objects a_n is then the *shift completion* of the one-object category $\mathcal{A}$ formed by the D-brane $a_0 = (\mathbf{L}_0, E, A)$ together with the morphism space $\mathrm{Hom}(a_0, a_0)$ and the induced morphism composition (figure 2). Indeed, the objects a_n can be identified with the *formal translates* $a_n = a[n]$ of the object $a_0 := a$, and the morphism spaces $\mathrm{Hom}(a_m, a_n) = \mathrm{Hom}(a[m], a[n])$

[14] That is, if none of the morphisms in the complex (8.8) vanishes.

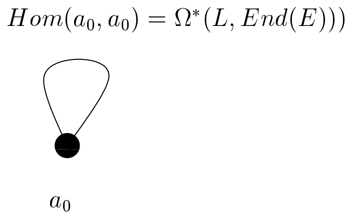

Figure 3: The open string field theory of [26] (identical with standard Chern-Simons field theory on L) corresponds to a one-object category $\mathcal{A}$, whose shift completion $\tilde{\mathcal{A}}$ gives the string field theory of the present paper, if one restricts to the shift-invariant case $E_n = E_0$ and $A_n = A_0$.

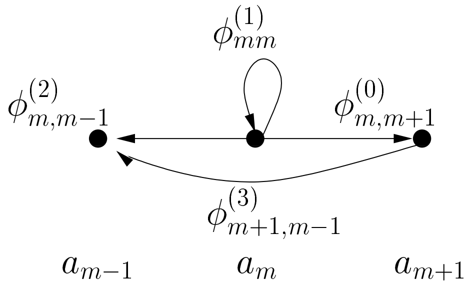

Figure 4: A pseudocomplex formed by three objects a_{m-1}, a_m and a_{m+1} and four morphisms $\phi_{m,m-1}$, $\phi_{m,m}$, $\phi_{m,m+1}$ and $\phi_{m+1,m-1}$ of $\tilde{\mathcal{A}}$.

are given by:

$$\mathrm{Hom}(a[m], a[n]) = \mathrm{Hom}(a, a)[n - m]\,. \tag{8.9}$$

The one-object category $\mathcal{A}$ describes the boundary sector of a single D-brane a_0, and underlies the open string field theory of [26], which is equivalent with the standard Chern-Simons field theory on the bundle $E_0 = E$. Indeed, since $m = n = 0$, the extra signs in the boundary product and topological metric of section 3 dissapear in this case. Hence *inclusion of graded D-branes a_n amounts to taking the shift completion $\tilde{\mathcal{A}}$ of the naive one-objects category $\mathcal{A}$*. That is, *working with graded D-branes amounts to taking the shift-completion.*

This is in fact a general principle, already pointed out in [13], which gives a more conceptual explanation for the observations of [3] and whose origin can be traced back to the theory of BV quantization. It is also intimately related with extended deformation theory [21, 22], as will be discussed in detail in [14].

In this categorical language, general solutions of (8.4) correspond to so-called *pseudocomplexes* [12] built out of objects and morphisms of the shift-completed category $\tilde{\mathcal{A}}$. For example, a solution of (8.4) containing only four non-vanishing components $\phi_{m,m-1}$, $\phi_{m,m}$, ϕ_{mm+1} and $\phi_{m+1,m-1}$ (for some fixed m) corresponds to the pseudocomplex depicted in figure 4.

It was shown on general grounds in [12, 13] that pseudocomplexes form a dG category $p(\tilde{\mathcal{A}})$ of their own and, in fact, give an admissible class of D-branes which extends the class of objects of $\tilde{\mathcal{A}}$. This allows for a string field theoretic description of any D-brane configuration resulting from condensation of string field components ϕ_{mn} satisfying eqs. (8.4). The resulting string field theory gives a description of open string dynamics in the presence of such general condensates, which include the original D-branes a_n, covariantly constant complexes of the type (8.8) as well as much more general objects, which do not admit a classical geometric description. *It is this category of "exotic A-type branes", rather than the naive one-object category $\mathcal{A}$ which must be studied in order to gain a better understanding of open string mirror symmetry "with one cycle".* This gives a very non-trivial extension of the theory originally considered in [26], and provides a first step toward an open string realization of the program outlined in [28] of gaining a better understanding of mirror symmetry by considering the *extended* moduli space of topological strings.

8.3.2 Relation with work of Bondal and Kapranov

As pointed out in [12, 13], pseudocomplexes over $\tilde{\mathcal{A}}$ allow one to make contact with the so-called *enhanced triangulated categories* discussed by Bondal and Kapranov [17]. In fact, it is easy to see that pseudocomplexes over $\tilde{\mathcal{A}}$ can be identified with the *twisted complexes* of [17], *defined over* $\mathcal{A}$. The latter form a dG category, the so-called pre-triangulated category Pre-Tr($\mathcal{A}$) associated with the one-object category $\mathcal{A}$. Hence we have the identification:

$$p(\tilde{\mathcal{A}}) = \text{Pre-Tr}(\mathcal{A})\,. \tag{8.10}$$

This is a reflection of the general principle, already mentioned in [13], that *the category of pseudocomplexes of the shift-completion $\tilde{\mathcal{A}}$ coincides with the pre-triangulated category* Pre-Tr($\mathcal{A}$) *of the uncompleted category $\mathcal{A}$.* From this point of view, the appearance of pre-triangulated categories (and, later, of triangulated categories) in string theory is a result of taking the shift-completion. This is the general formulation of the main observation made by M. Douglas in [3].

Upon taking the zeroth BRST cohomology (which, due to the existence of shift functors, corresponds to working *on-shell*) one obtains the so-called *enhanced triangulated category* $\mathrm{Tr}(\mathcal{A}) = H^0(\text{Pre-Tr}(\mathcal{A}))$ of the original one-object category $\mathcal{A}$. As discussed in [17], the category $\mathrm{Tr}(\mathcal{A})$ behaves in a certain sense as a "derived category" of $\mathcal{A}$, thereby *providing an A-model analogue of the derived category picture familiar from studies of the B-model* [1, 3, 5]. This construction can be related to a degeneration of the large radius limit of the "derived category" of Fukaya's category [1, 35, 36, 37] (see section 9).

8.4 Generalized complexes and the quasiunitary cover

We saw above that the traditional approach to A-type brane dynamics (which largely consists of applying the results of [26]) must be extended in a rather non-trivial manner. We hope to have convinced the reader that, when confronted with the problem of analyzing the structure defined by the most general solutions of (8.4), the tools of category theory become not only directly relevant, but also unavoidable, at least as a first approach to organizing the resulting complexity.

It may then come as a surprise that the most general solutions of (8.4) *do not, in fact, suffice*. Indeed, it was shown in [12, 13] that the basic physical constraint of *unitarity* requires the consideration of even more general objects (the so-called *generalized complexes*) over $\mathcal{A}$, which can be described as "pseudocomplexes with repetition". In a generalized complex, one allows for a sequence a_{n_j} whose (non-necessarily distinct) terms belong to our D-brane family $\{a_n\}$, and one asks for solutions of the obvious generalization of (8.4). More precisely, a generalized complex over $\tilde{\mathcal{A}}$ can be described as a sequence (a_{n_j}) together with a family of morphisms $\phi_{n_i,n_j} \in \mathrm{Hom}^1(a_{n_i}, a_{n_j})$ subject to the conditions:

$$Q_{m_i,m_k}\phi_{m_i,m_k} + \sum_j \phi_{m_j,m_k}\phi_{m_i,m_j} = 0\,. \tag{8.11}$$

The generalization away from pseudocomplexes is due to the fact that we allow for repetitions $m_i = m_j$.

It was shown in [12] that generalized complexes automatically lead to a string field theory, the so-called *quasiunitary cover* $c(\tilde{\mathcal{A}})$ of the theory based on $\tilde{\mathcal{A}}$. The quasiunitary cover satisfies a minimal form of the physical constraint of unitarity. Namely, any condensation process in the string field theory based on $c(\tilde{\mathcal{A}})$ produces a D-brane which can be identified with an object of $c(\tilde{\mathcal{A}})$. Mathematically, this gives an extension of the Bondal-Kapranov theory, which is forced upon us due to very basic physical considerations. It is *this* theory which gives the ultimate (i.e. "physically closed") extension of the naive one-object theory described by $\mathcal{A}$.

9. Relation with the large radius limit of Fukaya's category

In the last section of this paper, I wish to give a short outline of the how the theory considered here may relate to the category constructed in [35, 36]. As I will discuss in more detail somewhere else, Fukaya's category is related to a "quantized" version of a category of *intersecting* A-type branes, where the quantum effects arise from disk instanton corrections to the semiclassical structure.

Since disk instanton effects are suppressed in the large radius limit, we can in first approximation consider the case when the Kahler class of our Calabi-Yau manifold belongs to the deep interior of the Kahler cone. In this case, it can be shown that the

A_∞ "category" considered in [35] reduces to a $\mathbf{Z}_2$-projection of a *substructure* of an associative differential graded category which satisfies the axioms of [12]. This substructure, which is not a category in the standard mathematical sense, only describes a *subsector* of the (large radius) open string field theory of the topological A-type string. More precisely, the construction proposed in [35, 36] corresponds to a "category" whose objects are A-type branes wrapping *distinct* and *transversely intersecting* lagrangian cycles, and whose morphisms are given by boundary condition *changing* states of the topological A-type string. This construction does not take into account the possibility of having different A-type branes wrapping the same lagrangian cycle (which is precisely the subject of interest in the present paper), nor does it consider the boundary sectors of strings starting and ending on the same D-brane. Moreover, the construction of Fukaya's categiory is currently largely performed in a $\mathbf{Z}_2$-graded approach, and should be extended by inclusion of $\mathbf{Z}$-gradings.

If we use the notation $supp(a)$ to describe the support of a topological A-type brane, i.e. its underlying lagrangian cycle, then the large radius limit of Fukaya's category consists of a collection of objects having the property that $supp(a)$ and $supp(b)$ are transversely intersecting lagrangian cycles for any pair of *distinct* D-branes a and b. Moreover, the original proposal of [35] only considers morphism spaces of the form $\mathrm{Hom}(a, b)$ for $a \neq b$, i.e. no endomorphism spaces $\mathrm{End}(a) := \mathrm{Hom}(a, a)$ are allowed. Due to this reason, the resulting large radius structure does not correspond to a category in the classical mathematical sense, in spite of the fact that all off-shell nonassociativity can be eliminated in the large radius limit. For example, this structure does not contain units $1_a \in \mathrm{Hom}(a, a)$, since morphism spaces of the type $\mathrm{Hom}(a, a)$ are not directly described. This leads to somewhat complicated constructions [36, 37] which attempt to repair such problems by making use of transversality arguments.[15]

It is now clear that the theories described in this paper belong precisely to the missing sector of the original construction of [35, 36]. Indeed, we studied exactly the case when one has distinct topological D-branes which wrap the same (special) lagrangian cycle. One can view part of this as a certain degeneration of the large radius version of Fukaya's "category" in which various lagrangian cycles are deformed until they coincide. More precisely, it seems likely that only the sector of unidirectional off-diagonal deformations discussed in subsection 8.2. can be recovered in this

[15]The main idea of this approach is to view the Fukaya category as a description of a certain form of intersection theory for lagrangian cycles. In this case, the missing endomorphisms can be recovered by considering lagrangian *self*-intersections, which can be defined in traditional topological manner as intersections between a lagrangian cycle and a small displacement of itself (in this symplectic theory, such a displacement corresponds to an isotopy transform of the cycle). While one expects such a procedure to be related to the physical approach of including diagonal boundary sectors from the very beggining (or at least to a subsector of the physical construction), a reasonably complete proof that this is indeed the case has not yet been given. I thank Prof. K. Fukaya for some clarifications on the current status of his work on this issue.

manner from Fukaya's category. However, the issue is clouded by that fact that, in a string field theory such as those discussed in [12, 13], physics is invariant with respect to so-called *quasiequivalences*, and for this (as well and other) reasons the issue is currently unsettled. A perhaps more natural point of view is to follow the physics by including such objects in the very definition of the relevant category, as required by the structure of open string field theory.

10. Conclusions and directions for further research

We studied (the large radius limit of) a sector of the string field theory of the open A-model, by considering a system of distinct topological D-branes which wrap the same special lagrangian cycle. We extracted the relevant string field action from first physical principles, and identified it with a $\mathbf{Z}$-graded version of super-Chern-Simons field theory, thereby relating graded A-brane dynamics with the mathematical theory of $\mathbf{Z}$-graded superconnections developed in [19].

Upon using the resulting string field action, we gave a preliminary discussion of the associated moduli space of vacua, and we sketched its relation it with the theory of *extended* deformations [21, 22] of flat connections on the cycle. Moreover, we studied the effect of vacuum deformations form the perspective of [12, 13], which relates them to condensation of boundary and boundary condition changing operators and to formation of D-brane composites. This gives an explicit realization of the general discussion of those papers and shows the existence of a large class of "exotic" A-type branes. We also made a few observations about the connection of this physically motivated construction with the theory of enhanced triangulated categories developed in [17].

Our analysis should be viewed as a description of a small sector of the A-model counterpart of the "derived category of D-branes" whose B-model incarnation leads to the derived category of coherent sheaves. It is a basic consequence of mirror symmetry for open strings that the processes of D-brane composite formation which are responsible for generating the derived category of coherent sheaves in the B-model should have an A-model counterpart. Just as D^bCoh can be viewed as the product of off-shell B-type open string dynamics [3, 5, 13, 14, 16], A-model composite formation processes lead to an enlargement of the standard category of A-type topological D-branes. This enlargement, which after quantization (= inclusion of disk instanton effects) can be viewed as a sort of "derived category" of a completed[16] version of Fukaya's category, generates the A-model counterpart of D^bCoh.[17]

The generalized topological D-brane composites constructed in this paper correspond to backgrounds belonging to the extended moduli space of the topological

[16] Completed by inclusion of endomorphisms.

[17] Such an enlargement of Fukaya's category was already proposed in [1], though the proposal of that paper does not consider diagonal boundary sectors.

A-string. As such, they should be relevant for a better formulation of homological mirror symmetry, a subject which forms the deeper motivation of our study. It is important, however, to approach the much harder question whether such objects play a role in the physical, untwisted model. This issue could in principle be addressed through a careful study of deformations for the string field theory of compactified superstrings in the presence of graded A-type branes.

This problem is somewhat difficult to formulate precisely, due to the fact that current understanding of the *super*string field theory of Calabi-Yau compactifications is rather incomplete. Since the topological A-model only captures the chiral primary sector, it is in principle possible that some of the extended deformations leading to our exotic D-branes are "lifted" in the physical theory, due to the dynamics of higher string modes. It is likely, however, that at least the covariantly-constant complexes of subsection 8.2. survive in the untwisted model, since one can adapt effective action arguments such as those of [7] to argue for their appearance on non-topological grounds. In fact, these objects are the analogues of the type B D-brane complexes of [3, 5], and their relevance for Calabi-Yau *super*string dynamics should be similar to the importance of the latter. Whether the more exotic composites resulting from condensation of higher rank forms also play a role in the non-topological theory is currently an open problem, which deserves careful study.

Acknowledgments

I wish to thank Prof. Sorin Popescu for collaboration in a related project and Profs. D. Sullivan and K. Fukaya for useful conversations. I am endebted to Prof. M. Rocek for constant support and interest in my work. The author is supported by the Research Foundation under NSF grant PHY-9722101.

References

[1] M. Kontsevich, *Homological algebra of mirror symmetry*, proceedings of the International Congress of Mathematicians, Zurich, 1994, pp. 120–139, Birkhauser, [alg-geom/9411018].

[2] P. Seidel, *Graded lagrangian submanifolds*, *Bull. Soc. Math. France* **128** (2000) 103 [math.SG/9903049].

[3] M.R. Douglas, *D-branes, categories and $N = 1$ supersymmetry*, hep-th/0011017.

[4] M.R. Douglas, B. Fiol and C. Romelsberger, *Stability and BPS branes*, hep-th/0002037.

[5] P.S. Aspinwall and A.E. Lawrence, *Derived categories and zero-brane stability*, hep-th/0104147.

[6] A. Polishchuk and E. Zaslow, *Categorical mirror symmetry: the elliptic curve*, *Adv. Theor. Math. Phys.* **2** (1998) 443 [math.AG/9801119].

[7] Y. Oz, T. Pantev and D. Waldram, *Brane-antibrane systems on Calabi-Yau spaces*, *J. High Energy Phys.* **02** (2001) 045 [hep-th/0009112].

[8] M. Alishahiha, H. Ita and Y. Oz, *On superconnections and the tachyon effective action*, *Phys. Lett.* **B 503** (2001) 181 [hep-th/0012222].

[9] H. Ooguri, Y. Oz and Z. Yin, *D-branes on Calabi-Yau spaces and their mirrors*, *Nucl. Phys.* **B 477** (1996) 407 [hep-th/9606112].

[10] C.I. Lazaroiu, *On the structure of open-closed topological field theory in two dimensions*, *Nucl. Phys.* **B 603** (2001) 497 [hep-th/0010269].

[11] C.I. Lazaroiu, *Instanton amplitudes in open-closed topological string theory*, hep-th/0011257.

[12] C.I. Lazaroiu, *Generalized complexes and string field theory*, *J. High Energy Phys.* **06** (2001) 052 [hep-th/0102122].

[13] C.I. Lazaroiu, *Unitarity, D-brane dynamics and D-brane categories*, hep-th/0102183.

[14] C.I. Lazaroiu and S. Popescu, *String field theory, derived categories and extended moduli spaces*, to appear.

[15] C.I. Lazaroiu, *On the structure of topological sigma models with intersecting D-branes*, in preparation.

[16] D.-E. Diaconescu, *Enhanced D-brane categories from string field theory*, *J. High Energy Phys.* **06** (2001) 016 [hep-th/0104200].

[17] A. Bondal, M.M. Kapranov, *Enhanced triangulated categories*, *Mat. Sb.* **181** no. 5 (1990) 669, English translation in *Math. USSR Sbornik* **70** no. 1 (1991) 93.

[18] D. Quillen, *Superconnections and the Chern character*, *Topology* **24** no. 1. (1985) 89.

[19] J.M. Bismut and J. Lott, *Flat vector bundles, direct images and higher analytic torsion*, *J. Amer. Math. Soc.* **8** (1992) 291.

[20] C. Vafa, *Brane/anti-brane systems and* $\mathrm{U}(N|M)$ *supergroup*, hep-th/0101218.

[21] M. Kontsevich, *Deformation quantization of Poisson Manifolds 1*, q-alg/9709010.

[22] Yu.I. Manin, *Three constructions of Frobenius manifolds: a comparative study*, *Asian J. Math.* **3** no. 1 (1999) 179 [math.QA/9801006].

[23] Sergey Barannikov, *Generalized periods and mirror symmetry in dimensions* $n > 3$, math.AG/9903124.

[24] S. Axelrod and I.M. Singer, *Chern-Simons perturbation theory*, Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, vol. 1 New York, 1991, 3–45, World Sci. Publishing, River Edge, NJ, 1992 [hep-th/9110056].

[25] J.L. Cardy, *Conformal invariance in critical systems with boundaries*, Bonn 1986, Proceedings, *Infinite Lie algebras and conformal invariance in condensed matter and particle physics*, pp. 81–92; *Boundary conditions, fusion rules and the Verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581; *Boundary conditions in conformal field theory*, in *Integrable systems in quantum field theory and statistical mechanics*, M. Jimbo et al. eds., pp. 127–148;
J.L. Cardy and D.C. Lewellen, *Bulk and boundary operators in conformal field theory*, *Phys. Lett.* **B 259** (1991) 274.

[26] E. Witten, *Chern-simons gauge theory as a string theory*, in *The Floer memorial volume*, 637–678, Progr. Math., 133, Birkhauser, Basel, 1995 hep-th/9207094.

[27] E. Witten, *Topological sigma models*, *Comm. Math. Phys.* **118** (1988) 411.

[28] E. Witten, *Mirror manifolds and topological field theory*, in *Essays on mirror manifolds*, pp. 120–158, Internat. Press, Hong Kong, 1992 hep-th/9112056.

[29] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[30] C.B. Thorn, *String field theory*, *Phys. Rep.* **175** (1989) 1.

[31] B. Zwiebach, *Closed string field theory: quantum action and the B-V master equation*, *Nucl. Phys.* **B 390** (1993) 33 [hep-th/9206084].

[32] B. Zwiebach, *Oriented open-closed string theory revisited*, *Ann. Phys. (NY)* **267** (1998) 193 [hep-th/9705241].

[33] M.R. Gaberdiel and B. Zwiebach, *Tensor constructions of open string theories, 1. Foundations*, *Nucl. Phys.* **B 505** (1997) 569 [hep-th/9705038].

[34] A. Recknagel and V. Schomerus, *Moduli spaces of D-branes in CFT-backgrounds*, *Fortsch. Phys.* **48** (2000) 195 [hep-th/9903139]; *Boundary deformation theory and moduli spaces of D-branes*, *Nucl. Phys.* **B 545** (1999) 233 [hep-th/9811237]; *D-branes in Gepner models*, *Nucl. Phys.* **B 531** (1998) 185 [hep-th/9712186]; *D-branes in Gepner models*, *Nucl. Phys.* **B 531** (1998) 185 [hep-th/9712186];
N. Ishibashi, *The boundary and crosscap states in conformal field theories*, *Mod. Phys. Lett.* **A 4** (1989) 251;
T. Onogi and N. Ishibashi, *Conformal field theories on surfaces with boundaries and crosscaps*, *Mod. Phys. Lett.* **A 4** (1989) 161.

[35] K. Fukaya, *Morse homotopy, A^{∞}-category and Floer homologies*, in *Proceedings of the GARC Workshop on Geometry and Topology*, H.J. Kim ed., Seoul National University

1994, pp. 1–102;
Floer homology, A^∞-categories and topological field theory, in *Geometry and physics*, Lecture notes in pure and applied mathematics, **184** pp. 9–32, Dekker, New York, 1997; *Floer homology and mirror symmetry, 1*, preprint available at http://www.kusm.kyoto-u.ac.jp/~fukaya/fukaya.html.

[36] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory - anomaly and obstructon*, preprint available at http://www.kusm.kyoto-u.ac.jp/~fukaya/fukaya.html.

[37] M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, math.SG/0011041.